Club and Stationary Sets in ω_1, Probability Measures, and Ulam Matrices

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1 Introduction

In this talk I will discuss some interesting subsets of ω_1 – club-sets, Borel subsets, stationary and bi-stationary subsets of ω_1 – and the theorem about them that has been most useful to me over the years, called Fodor's Theorem. The question "How many types of stationary sets exist in ω_1?" will be a recurring theme. (Spoiler alert: there are uncountably many really distinct stationary subsets of ω_1.) In addition, I will mention some topological applications of stationary sets and Fodor's Lemma that have interested me over the years.

What do you need to know about ω_1 before we start? You need to know that ω_1 is an uncountable well-ordered set with the special property that if $\alpha < \omega_1$, then the initial segment $[0, \alpha)$ of ω_1 is countable, and that the countable union of countable sets is countable.

Like any linearly ordered set, ω_1 has an open interval topology. In ω_1, basic open neighborhoods of an ordinal α have the form $(\beta, \alpha+1) = (\beta, \alpha]$ where $\beta < \alpha$. As a result, if $\alpha = \beta+1$ is a successor ordinal, then the singleton $\{\alpha\} = (\beta, \alpha + 1)$ is open, and if $\alpha \neq 0$ is not a successor ordinal, then α is a limit point of the space ω_1.

The set of real numbers will be denoted by $\mathbb{R}$ and its usual linear order will be called $\prec$, while the usual ϵ-order on ω_1 will be denoted by $<$.

2 Closed unbounded subsets of ω_1

In the order-topology of ω_1, the uncountable closed subsets of ω_1 are very interesting. Clearly, a subset $S \subseteq \omega_1$ is uncountable iff it is cofinal in ω_1 iff it is unbounded in ω_1 so that uncountable closed subsets of ω_1 are called club-sets for "closed unbounded sets" [1].

Lemma 2.1 *For a subset $S \subseteq \omega_1$, S is a club-set if and only if there is a continuous bijection from ω_1 onto S.*

Proof: In case S is a club-set, the inductively defined order-preserving bijection $h : \omega_1 \to S$ is the desired bijection. Conversely, if there is a continuous bijection $g : \omega_1 \to S$, then S is certainly uncountable and therefore unbounded in ω_1. To show that S is a closed set, suppose λ is a limit

[1] sometimes abbreviated "CUB-sets"

point of S. Then there is a sequence $\sigma(n)$ of distinct points of S with $\sigma(n) \to \lambda$. For each n there is a unique $\alpha(n) \in \omega_1$ with $g(\alpha(n)) = \sigma(n)$. Then there is a subsequence[2] $\alpha(n_k)$ that converges to some $\beta \in \omega_1$. Then $g(\alpha(n_k)) \to g(\beta)$ because g is continuous. But then we have $g(\alpha(n_k)) = \sigma(n_k) \to \lambda$ and $g(\alpha(n_k) \to g(\beta)$ showing that $\lambda = g(\beta) \in S$. $\square$

Proposition 2.2 *If $\{C(n) : n < \omega\}$ is a countable family of club-sets, then $\bigcap\{C(n) : n < \omega\}$ is also a club-set.*

Proof: As in any topological space, any intersection of closed sets is closed, We use an interlacing argument to show that $\bigcap\{C(n) : n < \omega\}$ is nonempty and cofinal (= unbounded) in ω_1. Fix any $\alpha \in \omega_1$. There is a strictly increasing sequence $\alpha < \beta(1) < \beta(2) < \beta(3) < \cdots$ of elements of ω_1 where $\beta(n)$ is the first element in the n^{th} entry in the following list

$$C(1), C(2);\ C(1), C(2), C(3);\ C(1), C(2), C(3), C(4);\ C(1), \cdots, C(5);\ C(1), \cdots, C(6);\ C(1), \cdots$$

that is greater than $\alpha, \beta(1), \beta(2), \cdots, \beta(n-1)$. Then $\gamma = \sup\{\beta(n) : n < \omega\} \in \omega_1$ and will belong to each $C(k)$ because infinitely may terms of the convergent sequence $\langle \beta(n) \rangle$ belong to the closed set $C(k)$. $\square$

If you write down a subset of ω_1 as "the set of all elements of ω_1 with a certain specified property," chances are that you will describe a subset of ω_1 that contains a club-set, or whose complement contains a club-set. The collection of all such sets is a well-known class. Recall that $Borel(\omega_1)$ is the smallest σ-algebra of subsets of ω_1 that contains all closed subsets of ω_1.

Lemma 2.3 *A subset $S \subseteq \omega_1$ is a Borel set in ω_1 if and only if either S or $\omega_1 - S$ contains a club-set.*

Proof: Let $\mathcal{B} := \{S \subseteq \omega_1 : \text{either } S \text{ or } \omega_1 - S \text{ contains a club set}\}$. Then $\mathcal{B}$ is a σ-algebra containing all closed sets, so that $Borel(\omega_1) \subseteq \mathcal{B}$. We can complete the proof by showing that if the set $S \subseteq \omega_1$ contains some club-set C, then S is a Borel set. Because $S = C \cup (S - C)$, it will be enough to show that $S - C \in Borel(\omega_1)$.

Consider the open set $\omega_1 - C$. This open set is the union of a pairwise disjoint collection $\mathcal{D}$ of countable sets[3] of the form (α, β) with $\alpha < \beta \in \omega_1$. Then

$$(*) \quad S - C = S \cap (\omega_1 - C) = \bigcup\{S \cap D : D \in \mathcal{D} \textit{ and } S \cap D \neq \emptyset\}.$$

Suppose we choose one point $p(D) \in S \cap D$ whenever $D \in \mathcal{D}$ and $D \cap S \neq \emptyset$. Then the set $E := \{p(D) : D \in \mathcal{D} \textit{ and } S \cap D \neq \emptyset\}$ is the intersection of its own closure with the open set $\omega_1 - C$, so that the set E is a Borel set.

For each $D \in \mathcal{D}$ with $D \cap S \neq \emptyset$, index the set $S \cap D$ as $S \cap D = \{p(D, n) : n < \omega\}$, possibly with repetitions. By the previous paragraph, for each fixed n the set $E(n) := \{p(D, n) : D \in \mathcal{D} \textit{ and } D \cap S \neq \emptyset\}$ is a Borel set, and from (*) above, $S - C = \bigcup\{E(n) : n < \omega\}$, showing that $S - C$ is a countable union of Borel sets, and hence $S = C \cup (S - C)$ is also a Borel set. $\square$

[2] Any sequence in any linearly ordered set has a monotone subsequence

[3] These sets are the equivalence classes of the relation on $G = \omega_1 - C$ given by $\gamma \sim \delta$ iff every point between γ and δ belongs to G. Because C is cofinal in ω_1, each of the equivalence classes is countable.

Definition 2.4 *Suppose $\mathcal{B}$ is a σ-algebra of subsets of a set X. By a probability measure on $\mathcal{B}$ we mean a function $p : \mathcal{B} \to \mathbb{R}$ with the following properties:*

a) $p(B) \geq 0$ for all $B \in \mathcal{B}$;

b) $X \in \mathcal{B}$ and $p(X) = 1$

c) If B_n is a sequence of pairwise-disjoint sets in $\mathcal{B}$, then $p(\bigcup\{B_n : n < \omega\}) = \Sigma\{p(B_n) : n < \omega\}$.

If it happens that $p(B) \in \{0, 1\}$ for every $B \in \mathcal{B}$ then p is a two-valued probability measure.

Example 2.5 *Given Lemma 2.3 we obtain a two-valued probability measure on $Borel(\omega_1)$ if we define $p(B) = 1$ is B contains a club set, and $p(B) = 0$ otherwise.*

See the final section for more on probability measures.

3 Stationary sets in ω_1

One of the first really surprising results about ω_1 is that non-Borel subsets of ω_1 must exist (under AC). Mary Ellen Rudin gave the following elegant proof in [15]. In order to minimize the amount of choosing, my version of the proof is more cluttered than it would otherwise need to be.

Theorem 3.1 *There is a subset $S \subseteq \omega_1$ such that neither S nor $\omega_1 - S$ contains a club-set.*

Proof: Suppose not. Then for every subset $S \subseteq \omega_1$, either S or $\omega_1 - S$ contains a club-set. Because $|\omega_1| \leq |\mathbb{R}|$ we can fix an injection $g : \omega_1 \to [0, \to) \subseteq \mathbb{R}$.

For each integer $n \geq 1$ there is a collection $\mathcal{J}(n)$ of subsets of $[0, \to)$ satisfying

a) $\mathcal{J}(n)$ is countable and the usual ordering $\prec$ of $\mathbb{R}$ gives a well-ordering of each $\mathcal{J}(n)$;

b) $\bigcup \mathcal{J}(n) = [0, \to)$;

c) each $J \in \mathcal{J}(n)$ has diameter $\leq \frac{1}{n}$

For example, with $[a, b)$ denoting the usual half-closed interval in $(\mathbb{R}, \prec)$, we could let $\mathcal{J}(1) = \{[n, n+1) : n < \omega\}$ and $\mathcal{J}(n)$ be the collection of half-closed intervals with consecutive points in $\{k, k + \frac{1}{n}, k + \frac{2}{n}, \cdots : k < \omega, 1 \leq n \in \omega\}$ as endpoints. Each collection $\mathcal{J}(n)$ is welll-ordered.

Fix n. We claim that for some $J \in \mathcal{J}(n)$, the set $g^{-1}[J]$ contains a club-set in ω_1. If not, then for each $J \in \mathcal{J}(n)$, $\omega_1 - g^{-1}[J]$ contains a club-set C_J (apply our supposition that each subset of ω_1, or its complement, contains a club). By Proposition 2.2 the set $D := \bigcap\{C_J : J \in \mathcal{J}(n)\}$ is a club-set in ω_1. Consider any $\delta \in D$. Then $g(\delta) \in [0, \to)$ and yet for each $J \in \mathcal{J}(n)$, $\delta \in C_J \subseteq \omega_1 - g^{-1}[J]$ which shows that $g(\delta)$ does not belong to any $J \in \mathcal{J}(n)$ even though $\bigcup \mathcal{J}(n) = [0, \to)$. That is impossible, and so our claim is established.

Therefore, for each integer $n \geq 1$ we may choose the first $J_n \in \mathcal{J}(n)$ (in the well-ordering of $\mathcal{J}(n)$) such that $g^{-1}[J_n]$ contains some club-set D_n. Once again applying Proposition 2.2, we see that the set $E := \bigcap\{D_n : n \geq 1\}$ is a club-set, so we can choose α, β to be the first two members of E. Then $g(\alpha) \neq g(\beta)$ and for each n, $\alpha, \beta \in D_n \subseteq g^{-1}[J_n]$ so that for each integer $n \geq 1$ we have $0 < |g(\alpha) - g(\beta)| \leq diam(J_n) \leq \frac{1}{n}$ and that is impossible. Consequently, Theorem 3.1 is proved. □

Question: How much of the Axiom of Choice is needed in the proof of Theorem 3.1?

The set in Theorem 3.1 has a special property: even though it does not contain any club-set, it has a non-empty intersection with every club-set, because otherwise $\omega_1 - S$ would contain some club-set.

Definition: Any set that intersects every club-set in ω_1 is called a stationary subset of ω_1.

The set in Theorem 3.1 has a second property: its complement also intersects every club-set because otherwise S would contain a club-set.

Definition: Any set $S \subseteq \omega_1$ with the property that both S and $\omega_1 - S$ intersect every club-set is called a bistationary set.

Stationary and bistationary subsets of ω_1 are "big" sets and behave almost like second category subsets of $\mathbb{R}$ as our next result shows.

Corollary 3.2 *If $S \subseteq \omega_1$ is stationary and $S = \bigcup\{A_n : n \geq 1\}$, then some set A_n is stationary.*

Proof: Otherwise for each n there would be a club-set C_n with $A_n \cap C_n = \emptyset$. By Proposition 2.2, the set $D := \bigcap\{C_n : n \geq 1\}$ is a club-set. But $D \cap A_n = \emptyset$ for each n so that $D \cap S = \emptyset$, and that is impossible because S is stationary. □

In my own work, the most important property of stationary subsets has been a result known as "Fodor's Theorem" or the "Pressing Down Lemma" (PDL) concerning what are called pressing-down functions.

Definition: For a subset $S \subseteq \omega_1$, any function $f : S \to \omega_1$ with $f(\alpha) < \alpha$ for each $\alpha \in S - \{0\}$ is a pressing-down function. Such functions are also known as regressive functions. (See [4, 12].)

Theorem 3.3 *(Fodor's Theorem) Suppose S is a stationary subset of ω_1 and suppose $f : S \to \omega_1$ satisfies $f(\alpha) < \alpha$ for each $\alpha \in S - \{0\}$. Then f is constant on a stationary subset, i.e., there is some $\beta \in \omega_1$ and some stationary set T of ω_1 with $T \subseteq S$ and having $f(\alpha) = \beta$ for all $\alpha \in T$.*

The usual proof of Fodor's theorem uses an idea called "diagonal intersection" of club-sets as in the next lemma.

Lemma 3.4 *Suppose $D(\alpha)$ is a club-set for each $\alpha < \omega_1$. Then the set $E := \{\delta < \omega_1 : \delta \in \bigcap\{D(\alpha) : \alpha < \delta\}\}$ is a club-set.*

Proof: Replacing $D(\alpha)$ by the club-set $\bigcap\{D(\beta) : \beta \leq \alpha\}$ if necessary, we may assume that $D(\beta) \subseteq D(\alpha)$ whenever $\alpha < \beta$.

First we show that the set E is cofinal in ω_1. Start with any $\alpha(0) < \omega_1$ and choose $\alpha(1) \in D(\alpha(0))$ with $\alpha(0) < \alpha(1)$. Choose $\alpha(2) \in D(\alpha(1))$ with $\alpha(0) < \alpha(1) < \alpha(2)$. Inductively define $\alpha(n)$ so that $\alpha(0) < \alpha(1) < \alpha(2) < \cdots < \alpha(n) < \alpha(n+1)$ and $\alpha(n+1) \in D(\alpha(n))$. Compute $\gamma = \sup\{\alpha(n) : n < \omega\}$. Observe that if $m < n$ then $\alpha(n) \in D(\alpha(n-1)) \subseteq D(\alpha(m))$. Because $D(\alpha(m))$ is closed, we know that $\gamma \in D(\alpha(m))$. Therefore $\gamma \in \bigcap\{D(\alpha(m)) : m < \omega\} = \bigcap\{D(\alpha) : \alpha < \gamma\}$, showing that $\gamma \in E$ and $\gamma > \alpha(0)$.

Next we show that E is closed. Suppose λ is a limit point of E. Then there is a strictly increasing sequence sequence $\delta(n) \in E$ with $\lambda = \sup\{\delta(n) : n < \omega\}$. For each fixed $m < \omega$, if $n > m$ then $\delta(n) \in \bigcap\{D(\alpha) : \alpha < \delta(n)\} \subseteq \bigcap\{D(\alpha) : \alpha < \delta(m)\}$. Because $\bigcap\{D(\alpha) : \alpha < \delta(m)\}$ is closed, it follows that $\lambda \in \bigcap\{D(\alpha) : \alpha < \delta(m)\}$. Because $m < \omega$ was arbitrary, we know that

$$\lambda \in \bigcap\{\bigcap\{D(\alpha) : \alpha < \delta(m)\} : m < \omega\} = \bigcap\{D(\alpha) : \alpha < \lambda\}$$

so that $\lambda \in E$, as required to show that E is closed. □

Now we can prove Fodor's Theorem. We have a stationary set S and a pressing-down function $f : S \to \omega_1$. For contradiction, suppose no fiber $f^{-1}[\beta]$ of f is stationary. For each $\alpha \in S$ the set $N(f(\alpha)) = f^{-1}[f(\alpha)]$ is non-stationary, so there is a club-set $D(f(\alpha))$ with $N(f(\alpha)) \cap D(f(\alpha)) = \emptyset$. For each $\alpha \notin \{f(\beta) : \beta \in S\}\}$ let $D(\alpha) = \omega_1$. Then the set $E = \{\delta < \omega_1 : \delta \in \bigcap\{D(\beta) : \beta < \delta\}\}$ is a club-set. Because S is stationary and the set E is a club-set, there is some $\delta \in S \cap E$. Because $\delta \in S$ we know that $f(\delta) < \delta$ and that the non-stationary set $N(f(\delta))$ and the club-set $D(f(\delta))$ are defined and have $N(f(\delta)) \cap D(f(\delta)) = \emptyset$. In addition, we see that $\delta \in f^{-1}[f(\delta)] = N(f(\delta))$, and because $\delta \in E$ and $f(\delta) < \delta$ we also have $\delta \in \bigcap\{D(\alpha) : \alpha < \delta\} \subseteq D(f(\delta))$ contrary to $N(f(\delta)) \cap D(f(\delta)) = \emptyset$. This proves Fodor's Theorem. □

4 Topological properties of stationary subsets of ω_1

The proof of our next result includes an example of what one must do to exploit the definition of a stationary set.

Proposition 4.1 *Suppose $S \subseteq \omega_1$ is a stationary set and suppose $f : S \to \mathbb{R}$ is a continuous function. Then f is eventually constant, i.e., there is some $\alpha \in \omega_1$ with the property that $f(\alpha) = f(\beta)$ whenever $\alpha < \beta \in S$. Consequently, the set $\{f(\gamma) : \gamma \in \omega_1\}$ is countable.*

Proof: We start with an almost-proof and then show how to make it a real proof. Our first step is to prove that if $1 \leq m$ is given, then there is some α_m with the property that whenever $\beta, \gamma \in S \cap [\alpha_m, \omega_1)$, then $|f(\beta) - f(\gamma)| < \frac{1}{m}$. Once we have such α_m, we would let $\alpha = sup\{\alpha_n : 1 \leq n < \omega\}$ and we would know that $\alpha \in \omega_1$ and that if $\alpha < \beta, \gamma \in S$, then $f(\beta) = f(\gamma)$ as required.

Now fix $m \geq 1$ and we will almost succeed in finding α_m with the properties described above. Suppose no such α_m exists and let β_1 be the first point of S. This allows us to choose a sequence of points of S having $\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \cdots$ with $|f(\beta_k) - f(\gamma_k)| \geq \frac{1}{m}$ for each k. Compute $\delta = \sup\{\beta_k : 1 \leq k < \omega\}$. Clearly $\delta = \sup\{\gamma_k : 1 \leq k < \omega\}$ so that we have

$$0 = |f(\delta) - f(\delta)| = |\lim(f(\beta_k)) - \lim(f(\gamma_k))| = \lim |f(\beta_k) - f(\gamma_k)| \geq \frac{1}{m}$$

and that is impossible. But – and here is the problem – how do we know that $f(\delta)$ is defined, i.e., how do we know that $\delta \in S$?

Fixing that problem involves a standard trick. Let us say that an ordinal $\eta < \omega_1$ is f-ok if there is a sequence $\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \cdots$ in S having $|f(\beta_k) - f(\gamma_k)| \geq \frac{1}{m}$ and $\eta = \sup\{\beta_k : 1 \leq k < \omega\} = \sup\{\gamma_k : 1 \leq k < \omega\}$. Show that the set $E = \{\eta : \eta \text{ is } f - ok\}$ is a club-set, and once we know that, then there is some $\delta \in E \cap S$ that is the supremum of sequences β_k and γ_k as in the first paragraph. Now the proof is complete.□

We will illustrate the utility of Lemma 3.3 by giving a few of its topological consequences.

Proposition 4.2 *Suppose S is a stationary subset of ω_1 and suppose that $f : S \to \omega_1$ is a continuous*[4] *function with the property that for each $\beta \in \omega_1$, the set $f^{-1}[\beta]$ is countable. Then the set $T = \{f(\alpha) : \alpha \in S\}$ is also a stationary subset of ω_1.*

Proof: Suppose T is not stationary. Then there is a club-set C with $T \cap C = \emptyset$ so that $T \subseteq \omega_1 - C$. The set $\omega_1 - C$, like any open subset of ω_1, breaks into a union of pairwise disjoint open intervals called convex components of $\omega_1 - C$.[5] Let $\mathcal{V}$ be the collection of all convex components of $\omega_1 - C$. Because C is cofinal in ω_1, each $V \in \mathcal{V}$ must be countable.

Let $\mathcal{U} := \{f^{-1}[V] : V \in \mathcal{V}\}$. Then $\mathcal{U}$ is a pairwise disjoint collection of relatively open subsets of S, and because each fiber $f^{-1}[\beta]$ of f is countable, each set $f^{-1}[V]$ is also countable.

It is easy to prove that because S is a stationary subset of ω_1, then so is the set S^d consisting of all limit points of S that belong to S, and we have $S^d \subseteq S \subseteq \bigcup \mathcal{U}$. (Warning: S^d is not the same as the set of limit ordinals that happen to belong to S.) For each $\alpha \in S^d$ choose the unique member $U_\alpha \in \mathcal{U}$ with $\alpha \in U_\alpha$. Because α is a limit point of S, there is a point $g(\alpha) < \alpha$ such that $[g(\alpha), \alpha] \cap S \subseteq U_\alpha$. Then $g : S^d \to \omega_1$ is a pressing-down function with the stationary set S^d as its domain, so there must be some $\beta \in \omega_1$ for which $g^{-1}[\beta]$ is uncountable. Let α_1 be the first member of $g^{-1}[\beta]$ and choose $U_1 \in \mathcal{U}$ such that $\alpha_1 \in U_1$. The set U_1 is countable, so there must be some $\alpha_2 \in g^{-1}[\beta]$ that is strictly above every point of U_1. Therefore there is some $U_2 \in \mathcal{U}$ with $\alpha_2 \in U_2$ and $U_1 \neq U_2$. Therefore $U_1 \cap U_2 = \emptyset$. But $\alpha_1 \in [\beta, \alpha_2] \cap S \subseteq U_2 \cap S$ and $\alpha_1 \in U_1$ so $U_1 \cap U_2 \neq \emptyset$. That contradiction shows that the set $T = g[S]$ must be stationary, as claimed. □

And now let's turn to the question that interested me most when I learned about stationary sets: How many different types of stationary subsets of ω_1 exist? That depends on what "same" and "different" mean. Recall that two topological spaces X and Y are <u>homeomorphic</u> provided there is a bijective function $h : X \to Y$ with the property that both h and h^{-1} are continuous. If $S \subseteq \omega_1$ is a stationary set that contains a club-set and T is a stationary set that does not, then it is an easy matter to prove that S and T cannot be homeomorphic (because S has an uncountable subspace that contains the limit of each of its sequences, while T does not). But what if neither S nor T contains a club? In the most extreme case, what if $S \cap T = \emptyset$ as in Theorem 3.1? Could such sets be homeomorphic to each other?

Proposition 4.3 *[5] Suppose S and T are stationary subsets of ω_1 and that $S - T$ is stationary (e.g., in case $S \cap T = \emptyset$). Then there cannot be a continuous injective mapping from S into T, so that S and T cannot be homeomorphic.*

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[4]Recall that $f : S \to \omega_1$ is <u>continuous</u> provided for each open set $H \subseteq \omega_1$, the set $f^{-1}[H]$ is a relatively open subset of S, i.e., that $f^{-1}[H] = S \cap G$ for some open subset G of ω_1.

[5]The <u>convex components</u> of any open set W are the equivalence classes of the relation given by $\alpha \sim \beta$ iff $Conv(\alpha, \beta) \subseteq W$, where $Conv(\alpha, \beta) = [\alpha, \beta]$ if $\alpha \leq \beta$ and $Conv(\alpha, \beta) = [\beta, \alpha]$ if $\beta \leq \alpha$.

Proof: For contradiction, suppose that $S - T$ is stationary and there is a continuous injective mapping $h : S \to T$. Break S into three subsets

$$A := \{\alpha \in S : h(\alpha) < \alpha\}$$
$$B := \{\alpha \in S : h(\alpha) = \alpha\}$$
$$C := \{\alpha \in S : \alpha < h(\alpha)\}.$$

Note that $B \subseteq S \cap T \subseteq T$ so that $B \cap (S - T) = \emptyset$. Therefore $S - T \subseteq S = A \cup B \cup C$ gives $S - T \subseteq A \cup C$. Because $S - T$ is stationary, Corollary 3.2 yields that either A is stationary, or else C is stationary.

If the set A were stationary, we would have a violation of Lemma 3.3 because $h|_A$ is a one-to-one, pressing-down function. If the set C is stationary, then so is $h[C]$, by Proposition 4.2. But then the function h^{-1} restricted to the stationary set $h[C]$ would violate the Pressing Down Lemma (Theorem 3.3). □

Corollary 4.4 *Suppose there is a continuous injective mapping $h : S \to T$ where S and T are stationary sets. Then $S \cap T$ is also stationary.*

Proof: Because there is a continuous injective mapping from S to T, Proposition 4.3 shows that $S - T$ cannot be stationary. But $S = (S \cap T) \cup (S - T)$ is stationary so that the set $S \cap T$ must be stationary.□

Proposition 4.3 and Corollary 4.4 show that the key to the existence of a homeomorphism between stationary sets S and T is the nature of $S - T$ and $S \cap T$. For a fuller discussion, see [5].

More recent work has extended Proposition 4.3 in surprising ways. For any topological space X, the set of continuous real-valued functions on X is denoted by $C(X)$. There are many reasonable topologies that one might use for $C(X)$, and one of them is the "topology of pointwise convergence." In this topology, basic neighborhoods of a function $g \in C(X)$ are specified by a finite set $F \subseteq X$ and a positive real number ϵ and have the form $N(g, F, \epsilon) := \{h \in C(X) : |g(x) - h(x)| < \epsilon \text{ for all } x \in F\}$; see [6]. We indicate that the pointwise convergence topology is being used by writing $C_p(X)$. This space $C_p(X)$ is usually not metrizable, but it is a locally convex topological vector space and its properties are determined by the topological properties of X. The next result is due to R. Buzyakova in [2]:

Proposition 4.5 *Suppose S and T are stationary sets in ω_1 such that $S - T$ is stationary. Then there cannot exist any continuous, one-to-one function from $C_p(T)$ into $C_p(S)$.* □

The next result follows from Lemma 3.3. For each limit ordinal $\lambda \in \omega_1$, we can choose an increasing sequence $\alpha(\lambda, 1) < \alpha(\lambda, 2) < \alpha(\lambda, 3) < \cdots$ whose supremum is λ. Perhaps surprising, this cannot be done in any uniform way, as the next result shows. We leave the proof to readers who want to exercise their Pressing Down skills.

Corollary 4.6 *Suppose for each non-zero limit ordinal $\lambda \in \omega_1$, we have a strictly increasing sequence $\langle \alpha(\lambda, n) \rangle$ whose supremum is λ. It is not possible that $\alpha(\lambda, n) \leq \alpha(\mu, n)$ for all non-zero limit ordinals λ, μ with $\lambda < \mu$ and for all $n < \omega$.* □

Recall that a cardinal κ is regular if the cofinality of κ equals κ, i.e., κ is not the supremum of fewer, smaller cardinals. Almost everything we have said about ω_1 and its stationary sets, including the PDL, is true with small variations for any uncountable regular cardinal but not always for other cardinals (for example, the Pressing Down Lemma fails for $\omega_\omega = \sup\{\omega_n : n < \omega\}$). We give two final examples of the role that stationary sets play in my kind of topology. Recall that any linearly ordered set has an open-interval topology, and when endowed with the topology, the set becomes a linearly ordered topological space or LOTS. Any LOTS is a good space in terms of elementary topology, being Hausdorff, regular, completely regular, normal, and hereditarily normal. Often the first hard question about a LOTS is whether it is paracompact[6], and stationary subsets of regular uncountable ordinals are the key.

Proposition 4.7 *[7] Suppose X is a LOTS. Then X fails to be paracompact if and only if there is a stationary subset S of a regular uncountable cardinal that embeds as a closed subset of X.*

Using a remarkable generalization of Proposition 4.7 by Balogh and Mary Ellen Rudin [1], Buzyakova and Vural [3] proved that

Proposition 4.8 *Any monotonically normal[7] topological group is paracompact. In particular, any LOTS that is a topological group is paracompact.*

5 Ulam matrices, probability measures, and pairwise disjoint stationary sets

In this section we return to the question "How many different stationary sets can ω_1 have?" We show that there are uncountably many pairwise disjoint stationary sets in ω_1 using ideas that S. Ulam used to solve a problem in measure theory. See [13] for an extended discussion.

Recall the definition of a *probability measure* given in an earlier section. A probability measure on a set X consists of two things, namely, a collection $\mathcal{A}$ of subsets of X and a function $p : \mathcal{A} \to [0, 1]$ satisfying

(i) $X \in \mathcal{A}$ with $p(X) = 1$; and

(ii) the collection $\mathcal{A}$ is a σ-algebra, i.e., $\mathcal{A}$ is closed under the formation of countable unions and complements; and

(iii) the function p is countably additive, i.e.,

$$p(\bigcup\{A_n : n < \omega\}) = \Sigma\{p(A_n) : n < \omega\}$$

whenever $\langle A_n : n < \omega\rangle$ is a sequence of pairwise disjoint members of $\mathcal{A}$.

[6] A space X is paracompact if every open cover of X has an open, locally refinement, equivalently, if any open cover of X has a partition of unity subordinate to it.

[7] In a space X let $Pairs = \{(A, U) : A \subseteq U \subseteq X, A$ is closed and U is open$\}$. Then X is monotonically normal if for each $(A, U) \in Pairs$ there is an open set $G(A, U)$ with $A \subseteq G(A, U) \subseteq cl(G(A, U)) \subseteq U$ and having $G(A, U) \subseteq G(B, V)$ whenever $(A, U), (B, V) \in Pairs$ with $A \subseteq B$ and $U \subseteq V$. Any LOTS is monotonically normal [11].

A probability measure is *non-atomic* provided $p(\{x\}) = 0$ for each $x \in X$. Property (iii) shows that for any non-atomic probability measure, $p(C) = 0$ for any countable subset $C \subseteq X$ provided $\{x\} \in \mathcal{A}$ for each $x \in C$.

In an earlier section we gave an example of a non-atomic probability measure defined for all Borel subsets of ω_1.

Early in the last century, mathematicians thought about the question "Given a set X, how large can the domain of a non-atomic probability measure on X be? Could there be a non-atomic probability measure whose domain is the collection of all subsets of X?" In 1905 Vitali [17] showed that no translation-invariant probability measure on $\mathbb{R}$ or on $[0,1] \subseteq \mathbb{R}$ could be defined for all subsets of $\mathbb{R}$ (respectively, all subsets of $[0,1]$), and his proof is the standard approach in most analysis textbooks today ([10], [14]). However, what about probability measures that are not translation-invariant? In [16], Ulam took a different approach to the problem, starting with the uncountable well-ordered set ω_1. In that 1930 paper he introduced a combinatorial object now called an Ulam matrix [8] [13]. An <u>Ulam matrix</u> is a collection of subsets $\{U(n,\alpha) : n < \omega, \alpha \in \omega_1\}$ of ω_1 with three special properties. To remember the properties, it helps to display the Ulam matrix in a row and column format with countably many columns, one for each $n < \omega$, and uncountably many rows, one for each $\alpha \in \omega_1$.

$$\begin{pmatrix} \vdots & \vdots & & \vdots & \\ U(0,\alpha) & U(1,\alpha) & \cdots & U(n,\alpha) & \cdots \\ \vdots & \vdots & & \vdots & \\ U(0,1) & U(1,1) & \cdots & U(n,1) & \cdots \\ & & & & \\ U(0,0) & U(1,0) & \cdots & U(n,0) & \cdots \end{pmatrix}$$

a) each column is pairwise disjoint, i.e., for each fixed $n < \omega$ and distinct $\alpha, \beta \in \omega_1$, $U(n,\alpha) \cap U(n,\beta) = \emptyset$;

b) each row is pairwise disjoint, i.e., for each fixed $\alpha \in \omega_1$, if $n \neq m$, then $U(m,\alpha) \cap U(n,\alpha) = \emptyset$;

c) for each fixed $\alpha \in \omega_1$, $\bigcup\{U(n,\alpha) : n < \omega\} = \{\beta \in \omega_1 : \alpha < \beta\}$. (To simplify notation, we denote that last set by $(\alpha, \rightarrow)$.)

It is not clear that such matrices exist, and to define the sets $U(n,\alpha)$ we proceed as follows. For each $\gamma \in \omega_1$ the set $[0,\gamma)$ is finite or countable, so there is a one-to-one function $f_\gamma : [0,\gamma) \to [0,\omega)$. Define $U(n,\alpha) = \{\gamma : \alpha \prec \gamma \text{ and } f_\gamma(\alpha) = n\}$.

Fix $n < \omega$ and suppose $\gamma \in U(n,\alpha) \cap U(n,\beta)$. Then $\alpha, \beta \prec \gamma$ and $f_\gamma(\alpha) = n$ and $f_\gamma(\beta) = n$. Because f_γ is one-to-one, we know that $\alpha = \beta$. This proves (a).

Fix $\alpha \in \omega_1$ and suppose $\gamma \in U(n,\alpha) \cap U(m,\alpha)$. Then $\alpha \prec \gamma$ and $f_\gamma(\alpha) = m$ and $f_\gamma(\alpha) = n$. Because f_γ is a well-defined function, we see that $m = n$. This proves (b).

Finally, fix α. From the definition of $U(n, \alpha)$ we know that $\alpha < \gamma$ for each $\gamma \in U(n, \alpha)$, showing that $\bigcup\{U(n, \alpha) : n < \omega\} \subseteq (\alpha, \rightarrow)$. Next, consider any $\gamma \in (\alpha, \rightarrow)$. Then $\alpha \in [0, \gamma)$ so that $f_\gamma(\alpha) \in [0, \omega)$, say $f_\gamma(\alpha) = n$. But then $\gamma \in U(n, \alpha)$ so that we have $(\alpha \rightarrow) = \bigcup\{U(n, \alpha) : n \prec \omega\}$. This establishes (c) and we can now use our matrices to prove Ulam's theorem.

Theorem 5.1 *(Ulam) No non-atomic probability measure on ω_1 can be defined on $\mathcal{P}(\omega_1)$.*

Proof: Suppose there is a non-atomic probability measure defined for all subsets of ω_1. For each $\alpha \in \omega_1$ the set $[0, \alpha]$ is countable so that $p([0, \alpha]) = 0$. Hence $p((\alpha, \rightarrow)) = p(\omega_1) - p([0, \alpha]) = 1$. Because $(\alpha, \rightarrow) = \bigcup\{U(n, \alpha) : n < \omega\}$ by property (c), property (b) gives us that

$$1 = p((\alpha, \rightarrow)) = p(\bigcup\{U(n, \alpha) : n < \omega\}) = \Sigma\{p(U(n, \alpha)) : n < \omega\}$$

so that there must be some $n(\alpha) < \omega$ with $p(U(n(\alpha), \alpha) > 0$. Because ω_1 is uncountable while $[0, \omega)$ is countable, there must be an uncountable subset $A \subseteq \omega_1$ and a fixed $k \prec \omega$ with $n(\alpha) = k$ for all $\alpha \in A$. Hence $p(U(k, \alpha)) > 0$ for all $\alpha \in A$.

For each $\alpha \in A$ there is a positive integer $j(\alpha)$ with $p(U(k, \alpha)) > \frac{1}{j(\alpha)}$. Then there must be an uncountable $B \subseteq A$ and a fixed positive integer J with $j(\alpha) = J$ for each $\alpha \in B$. Choose $J + 1$ members of the set B, say $\alpha(1), \alpha(2), \cdots, \alpha(J + 1)$. The sets $U(k, \alpha(i))$ are pairwise disjoint by property (a) above so we must have

$$1 = p(\omega_1) \geq p(\bigcup\{U(k, \alpha(i)) : 1 \leq i \leq J + 1\}) = \Sigma\{p(U(k, \alpha(i)) : 1 \leq i \leq J + 1\} > (j + 1)\frac{1}{j} > 1$$

and that is impossible.□

Exercise: (a) Where was AC used in Ulam's theorem? (b) Let $\mathcal{U}$ be any free (= non-principle) ultrafilter on ω_1. Then for each subset $S \subseteq \omega_1$ we know that either $S \in \mathcal{U}$ or else $\omega_1 - S \in \mathcal{U}$. For $S \subseteq \omega_1$, define $p(S) = 1$ if $S \in \mathcal{U}$ and define $p(S) = 0$ otherwise. According to Theorem 5.1, this p is not a probability measure on the power set $\mathcal{P}(\omega_1)$. Why not? Now look up "real-valued measurable cardinal."

What could Ulam's theorem have to do with probability measures on more familiar spaces such as $[0, 1]$ or $\mathbb{R}$? If $|\omega_1| = |\mathbb{R}| = |[0, 1]|$ (i.e., if the Continuum Hypothesis holds) then there is a one-to-one function g from ω_1 onto $[0, 1]$ and the function g can be used to transfer the sets of the Ulam matrix into $[0, 1]$. The reason that we need the function g to be surjective is to insure that for each fixed α, the set $[0, 1] - g(\bigcup\{U(\alpha, n) : n < \omega\})$ is countable so that we can claim $p(\bigcup\{g(U(\alpha, n)) : n < \omega\}) = 1$. The best conclusion about $[0, 1]$ that we can get from Ulam's theorem is:

Theorem 5.2 *If the Continuum Hypothesis holds, then there is no non-atomic probability measure on $[0, 1]$ that is defined for all subsets of $[0, 1]$.*

We close with three consequences of the existence of Ulam matrices. Rudin's proof in Theorem 3.1 shows that we can get two disjoint stationary subsets of ω_1. In fact, one can get many pairwise disjoint stationary subsets of ω_1.

Corollary 5.3 *There are uncountably many pairwise disjoint stationary subsets of* ω_1.

Proof: For each fixed $\alpha \in \omega_1$ we know that $\bigcup\{U(n,\alpha) : n < \omega\}$ is the stationary set $(\alpha, \rightarrow)$. Therefore Corollary 3.2 assures us that there is some $n(\alpha) < \omega$ such that $U(n(\alpha),\alpha)$ is stationary. Because $[0,\omega)$ is countable while ω_1 is uncountable, there must be a $k < \omega$ and an uncountable $A \subseteq \omega_1$ such that $n(\alpha) = k$ for all $\alpha \in A$. Therefore, for $\alpha \in A$, the sets $U(k,\alpha)$ are all stationary and because they all lie in column number k of the Ulam matrix, they are pairwise disjoint by property (a). $\square$

From Proposition 4.3 we know that disjoint stationary subsets of ω_1 cannot be homeomorphic, so we have the following answer to our earlier question "How many different stationary sets exist in ω_1?"

Corollary 5.4 *There is an uncountable family of stationary subsets of* ω_1*, no two of which are homeomorphic to eachother.* $\square$

Recall Proposition 2.2: any countable intersection of club-sets is a club-set. Our final result shows how different club-sets and stationary sets can be.

Corollary 5.5 *There is a sequence* $\langle S_n : n < \omega \rangle$ *of stationary subsets of* ω_1 *with* $S_{n+1} \subseteq S_n$ *for each* n *and* $\bigcap\{S_n : n < \omega\} = \emptyset$.

Proof: From Corollary 5.4 we can find an infinite sequence $T_1, T_2, \cdots$ of pairwise disjoint stationary subsets of ω_1. Let $S_n := \bigcup\{T_k : n < k < \omega\}$. Then each S_n is stationary, and $\bigcap\{S_n : n \prec \omega\} = \emptyset$, as required. $\square$

References

[1] Balough, Z. and Rudin, M.E., Monotonic normality, Topology and its Applications 47(1992), 115-127.

[2] Buzyakova, R., Injections into function spaces over ordinals, *Topology and its Applications* 157(2010), 2844-2849.

[3] Buzyakova, R., and Vural, C., Stationary sets in topological and paratopological groups, Houston Journal of Mathematics 40(2014), 267-273.

[4] Devlin, K., *The Joy of Sets*, Springer-Verlag, New York, 1993.

[5] van Douwen, E., and Lutzer, D. On the classification of stationary sets, *Michigan Mathematics Journal* 26(1979), 47-64.

[6] Engelking, R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[7] Engelking, R. and Lutzer, D., Paracompactness in ordered spaces, *Fundamenta Mathematicae*, 94(1976), 49-58.

[8] Erdos, P., Ulam, the man and the mathematician, *Journal of Graph Theory* 9(1985), 445-449.

[9] Halmos, P., *Finite Dimensional Vector Spaces*, Springer-Verlag, New York, 1987.

[10] Halmos, P., *Measure Theory*, van Nostrand, New York, 1950.

[11] Heath, R., Lutzer, D., and Zenor, P., Monotonically normal spaces, *Transactions of the American Mathematical Society*, 178 (1973), 481–493.

[12] Kunen, K., *Set Theory: An Introduction to Independence Proofs*, North Holland, Amsterdam, 1980.

[13] Oxtoby, J., *Measure and Category*, Springer-Verlag, New York, 1971.

[14] Royden, H., *Real Analysis*, Macmillan, New York,1963.

[15] Rudin, M., A subset of the countable ordinals, *American Mathematical Monthly* 64(1957), 351-2.

[16] Ulam, S., Zur Masstheorie in der allgemeinen Mengenlehre, *Fundamenta Mathematicae* 16(1930), 140-150.

[17] Vitali, G., Sul problema della misura dei gruppi di punti di una retta, Nota, Bologna, 1905.